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Almost All Roots of $\zeta(s) = a$ Are Arbitrarily Close to $\sigma = 1/2$

(Riemann/zeta-function)

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ABSTRACT **Let $s = \sigma + it$. For any complex a, all but $O(1/\log\log T)$ of the roots of $\zeta(s) = a$ in $T < t < 2T$ lie in $|\sigma - 1/2| < (\log\log T)^2/\log T$. The results extend easily to other functions satisfying a functional equation such as the Dirichlet L-functions, the Lerch functions, etc.**

In his recent book (ref. 1, pp. 164 and 197) Edwards states that the clustering of the zeros of $\zeta(s)$ near $\sigma = 1/2$, first proved by Bohr and Landau (2), is the best existing evidence for the Riemann Hypothesis. Titchmarsh (ref. 3, p. 197) also emphasizes with italics the clustering phenomenon of the zeros of $\zeta(s)$. It will be shown here that for any complex a the roots of $\zeta(s) = a$ cluster at $\sigma = 1/2$ and so, in this sense, the case $a = 0$ is not special. However, from ref. 3, Chap. 11, it is clear that the clustering for the case $a = 0$ is more pronounced than for $a \neq 0$, since the roots of $\zeta(s) = a$, $a \neq 0$, in $\sigma \geq 1/2 + \delta$, $0 < t < T$, must exceed $K(a,\delta)T$, $K(a,\delta) > 0$, and so the results on $N(\sigma,T)$, such as those of Ingham (ref. 3, p. 203) and Selberg (ref. 3, p. 204), are peculiar to $a = 0$.

The clustering of the roots of $\zeta(s) = a$ at $\sigma = 1/2$ was demonstrated under the Riemann Hypothesis in 1913 by Landau (4).

THEOREM. *Let $T^{1/2} \leq U \leq T$. Let a be any fixed complex number. Let $N^{(1)}(a; T, U)$ be the number of roots of $\zeta(s) = a$ in*

$$\sigma > \frac{1}{2} + (\log\log T)^2/\log T,\ T < t < T + U;$$

let $N^{(2)}(a; T, U)$ be those in

$$\sigma < \frac{1}{2} - (\log\log T)^2/\log T,\ T < t < T + U;$$

and let $N^{(3)}(a; T, U)$ be those in

$$\frac{1}{2} - (\log\log T)^2/\log T \leq \sigma \leq \frac{1}{2} + (\log\log T)^2/\log T,\ T < t < T + U.$$

Then for large T

$$N^{(3)}(a; T, U) = \frac{U}{2\pi}\log T + O(U\log T/\log\log T)$$

$$N^{(1)}(a; T, U) + N^{(2)}(a; T, U) = O(U\log T/\log\log T).$$

LEMMA 1.

$$\int_T^{T+U}\left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = O(U\log T).$$

Proof: By the approximate functional equation

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \leq 2\Big|\sum_{n\leq(t/2\pi)^{1/2}} n^{-1/2-it}\Big| + O(t^{-1/4}).$$

By the familiar process of treating the diagonal and nondiagonal sums separately,

$$\int_T^{T+U}\Big|\sum_{n\leq(t/2\pi)^{1/2}} n^{-1/2-it}\Big|^2 dt = O(U\log T)$$

and the lemma is proved. [Far sharper results have been proved (ref. 3, Chap. 7).]

It will be assumed in what follows that $a \neq 1$. The minor modification for the case $a = 1$ will be indicated below.

LEMMA 2. *Denote the roots of $\zeta(s) = a$ by $\sigma_a + it_a$. Let b be real. Then*

$$2\pi\sum_{\substack{T<t_a<T+U\\ \sigma_a>-b}}(\sigma_a + b) = \int_T^{T+U}\log|\zeta(-b + it) - a|dt - U\log|1 - a| + O(\log T).$$

Proof: A familiar lemma of Littlewood (ref. 3, p. 187) is applied to $G(s) = (\zeta(s) - a)/(1 - a)$. If c is sufficiently large, it follows from integrating $\log G(s)$ around the boundary of the half-strip, $\sigma = c$, $T < t < T + U$; $\sigma > c$, $t = T$; $\sigma > c$, $t = T + U$, and taking the imaginary part of the result that

$$\int_T^{T+U}\log|G(c + it)|dt = O(1).$$

Use is made of $\log(1 + w) = O(w)$ for $|w| < 1/2$. Using Jensen's theorem in a familiar way (ref. 3, p. 180) in circles with centers at (c,T) and $(c,T + U)$ gives

$$\int_{-b}^{c}\arg G(\sigma + iT)d\sigma = O(\log T)$$

and similarly for $t = T + U$. This proves the lemma.

LEMMA 3.

$$2\pi \sum_{\substack{T<t_a<T+U \\ \sigma_a>1/2}} \left(\sigma_a - \frac{1}{2}\right) = O(U \log\log T)$$

Proof: Use *Lemma 2* with $b = -1/2$. Note that

$$\int_T^{T+U} \log\left|\zeta\left(\frac{1}{2}+it\right) - a\right| dt \le \frac{1}{2}U \times \log\left(\frac{1}{U}\int_T^{T+U}\left|\zeta\left(\frac{1}{2}+it\right) - a\right|^2 dt\right).$$

Application of *Lemma 1* gives the result.

LEMMA 4.

$$N^{(1)}(a; T, U) = O(U \log T/\log\log T).$$

Proof: From *Lemma 3* follows

$$2\pi N^{(1)}(a; T, U)(\log\log T)^2/\log T = O(U \log\log T)$$

and this proves the lemma.

LEMMA 5. *Let $b \ge 2$. Then for large T*

$$2\pi \sum_{T<t_a<T+U} (\sigma_a + b) = \left(\frac{1}{2}+b\right)\left[U \log\frac{T}{2\pi} - U + (T+U)\log\frac{T+U}{T}\right] - U\log|1-a| + O(\log T)$$

Proof: From the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2^s\pi^{-1+s} \sin \pi s/2\Gamma(1-s).$$

By Stirling's formula, $\chi(s) = \exp\left((\pi i/4) - 1 + f(s)\right)$ for $|\arg s - \pi/2| < \pi/4$ where

$$f(s) = \left(\frac{1}{2} - s\right)\log\frac{(1-s)i}{2\pi} + s + O(1/s).$$

Hence,

$$\log|\zeta(s) - a| = \log|\chi(s)| + \log|\zeta(1-s)| + O\left(\frac{a}{\chi(s)\zeta(1-s)}\right)$$

and

$$\log|\chi(s)| = \left(\frac{1}{2} - \sigma\right)\log|t/(2\pi)| + O(1/t).$$

Since $\chi(s)$ for $\sigma \le -2$ exceeds t^2 in size, there can be no zeros of $\zeta(s) - a = 0$ in $\sigma \le -2$ for large $|t|$. By *Lemma 2* the above formulae yield

$$2\pi \sum_{T<t_a<T+U} (\sigma_a + b) = \left(\frac{1}{2}+b\right) \times \int_T^{T+U} \log\frac{t}{2\pi}\,dt - U\log|1-a| + O(\log T)$$

and this proves the lemma.

LEMMA 6.

$$N^{(1)}(a; T, U) + N^{(2)}(a; T, U) + N^{(3)}(a; T, U) = \frac{U}{2\pi}\log\frac{T}{2\pi} - \frac{U}{2\pi} + \frac{T+U}{2\pi}\log\frac{T+U}{T} + O(\log T).$$

Proof: In *Lemma 5* subtract the case $b + 1$ from b and note there are no zeros in $\sigma \le -2$ for large T.

This result was proved by Landau (4). In particular, from *Lemma 6* follows

$$N^{(1)}(a; T, U) + N^{(2)}(a; T, U) + N^{(3)}(a; T, U) = \frac{U}{2\pi}\log T + O(U) \quad [1]$$

The use of Littlewood's lemma in the proof that follows is very similar to the use made in proving *Theorem 2* of ref. 5.

Proof of Theorem: From the definition of $N^{(j)}(a; T, U)$ and *Lemma 3* follows

$$2\pi \sum_{T<t_a<T+U} (\sigma_a + b) \le O(U \log\log T) + 2\pi\left(b + \frac{1}{2}\right)[N^{(1)} + N^{(3)}(a; T, U)] + 2\pi\left(b + \frac{1}{2} - (\log\log T)^2/\log T\right)N^{(2)}(a; T, U)$$

Using *Lemma 5* and *Lemma 6*, this yields

$$0 \le O(U \log\log T) - 2\pi(\log\log T)^2 N^{(2)}(a; T, U)/\log T$$

or

$$N^{(2)}(a; T, U) = O(U \log T/\log\log T).$$

Combining this with *Lemma 4* and [1] proves the theorem for $a \ne 1$.

For $a = 1$, let $G(s) = 2^s(\zeta(s) - 1)$ and make the minor changes necessary.

By Rouché's theorem and Stirling's formula for $\chi(s)$ it follows easily that there is also a root of $\zeta(s) - a = 0$ in the neighborhood of $s = -2n$ for large n, and with a finite number of possible exceptions, these are the only roots with $\sigma \le -2$.

A more general result than the theorem which has the same proof is the following:

Let $\delta > 0$ be small and $T^{1/2} \le U \le T$. For $T < t < T + U$, let $N_{(1)}(a, \delta; T, U)$ be the number of roots of $\zeta(s) = a$ in $\sigma > 1/2 + \delta$; let $N_{(2)}(a, \delta; T, U)$ be the number in $\sigma < 1/2 - \delta$; and let $N_{(3)}(a, \delta; T, U)$ be the number in $1/2 - \delta \le \sigma \le 1/2 + \delta$. Then

$$N_{(3)}(a, \delta; T, U) = \frac{U}{2\pi}\log T + O(U(\log\log T)/\delta)$$

$$N_{(1)} + N_{(2)}(a, \delta; T, U) = O(U(\log\log T)/\delta)$$

The theorem is the special case $\delta = (\log\log T)^2/\log T$.

P. T. Bateman has informed me that A. Selberg has written to him that he can sharpen the above results since he can prove that for $a \neq 0$

$$\int_0^T \log\left|\zeta\left(\frac{1}{2} + it\right) - a\right| dt \sim \frac{1}{2}\pi^{-1/2}T(\log\log T)^{1/2}.$$

This replaces *Lemma 3* and is used in *Lemma 2* with $b = -1/2$ to give the sharp result.

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